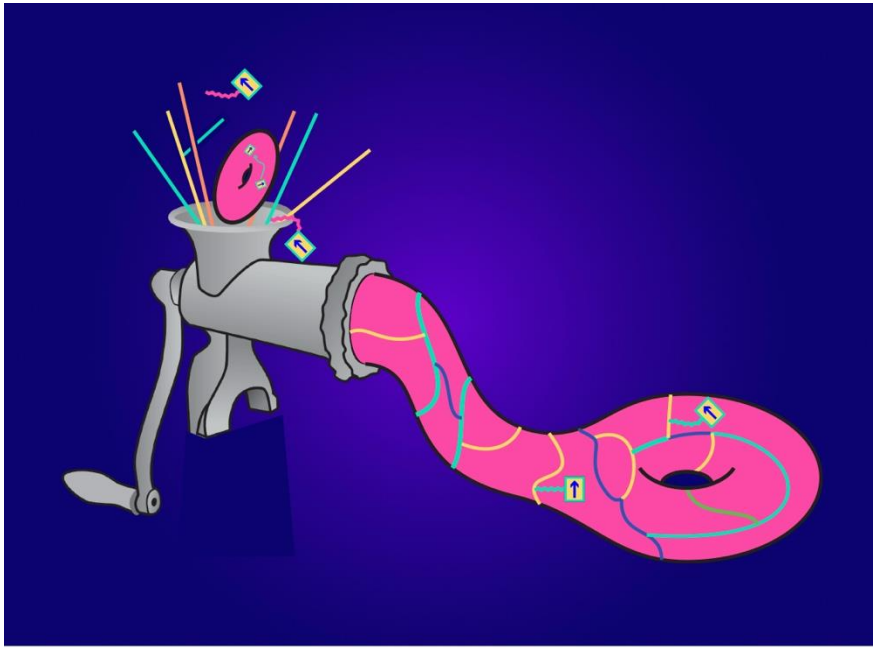
Topological Defects and Non-Invertible Symmetries 2

Paul Fendley

All Souls College, Oxford



Mainly based on work with David Aasen and Roger Mong (2016,2020) and Luisa Eck (2023,2024)

Outline for today

The uses of topological defects

Use #1: Generalized Kramers-Wannier duality

Use #2: Exact degeneracies for ground states and for kinks

Use #3: Non-obvious non-invertible mappings
(from XXZ to the Rydberg-blockade ladder)

Use #4: Exact lattice computation of g factor ratios and conformal spins

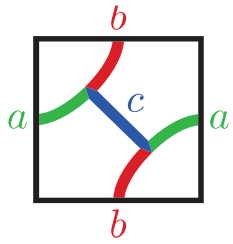
Including braiding gives

Use #5: Baxterization via terminating defect lines

Use #6: 2+1d models for chiral topological order

Fusion categories allow one to construct topological defects both on the lattice and in the continuum

The partition function in the presence of topological defects is independent of local deformations of the defects. Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:



Working in a Hilbert-space formalism, the defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Topological defects generate non-invertible/categorical/ generalized ``dualities'' and symmetries.

Manipulating the diagram

Data:

Non-negative integers N_{ab}^{c}

quantum dimensions d_a

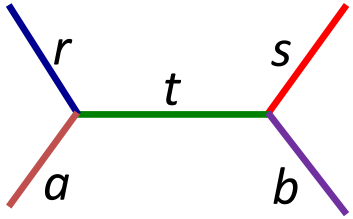
F symbols: $F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$

Isolated loop:

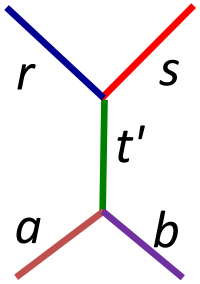


$= d_r$

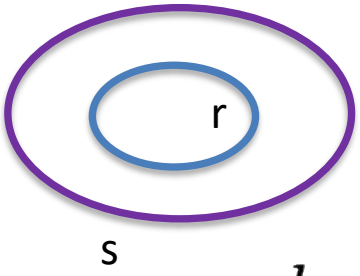
F move:



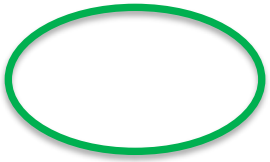
$$= \sum_{t' \in \mathcal{C}} F_{tt'} \begin{bmatrix} r & s \\ a & b \end{bmatrix}$$



Fusion:



$$= \sum_{t} N_{rs}^{t}$$

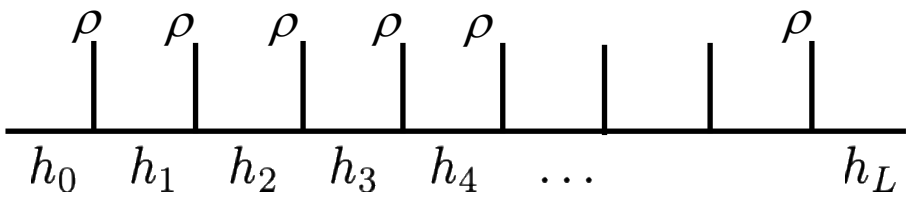


$$d_r d_s = \sum_{t} N_{rs}^{t} d_t$$

The Hilbert space

To define model, first pick a category C, and a particular object $\rho \in C$.

Each basis element of Hilbert space is then given by a fusion tree:



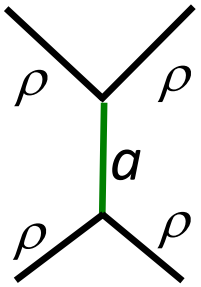
Degrees of freedom are "heights" $h_0, h_1, h_2, \ldots, h_L \in C$ obeying $N_{h_j \rho}^{h_{j+1}} = 1.$

(recall $a \otimes b = \sum_c N_{ab}^c \; c$)

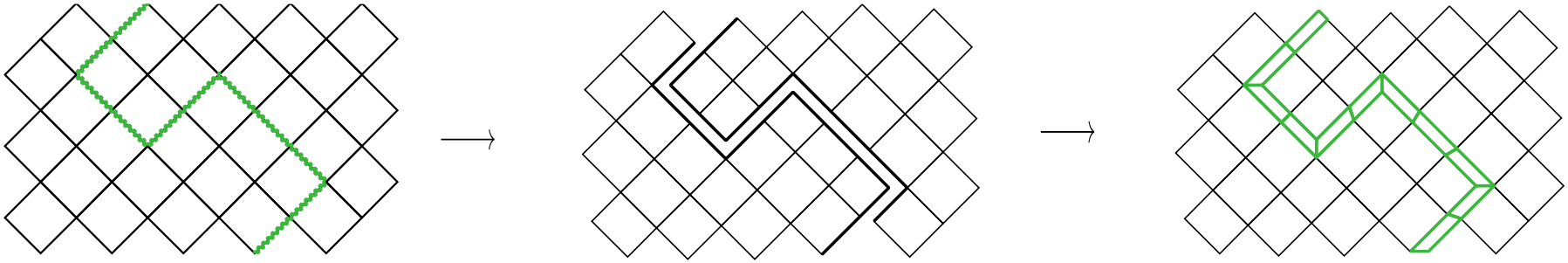
To get periodic boundary conditions, set $h_0 = h_L$.

Hamiltonian is any combination of projection operators

$P^{(a)} \propto$



Inserting a defect



The defects have a weight depending on the adjacent heights:

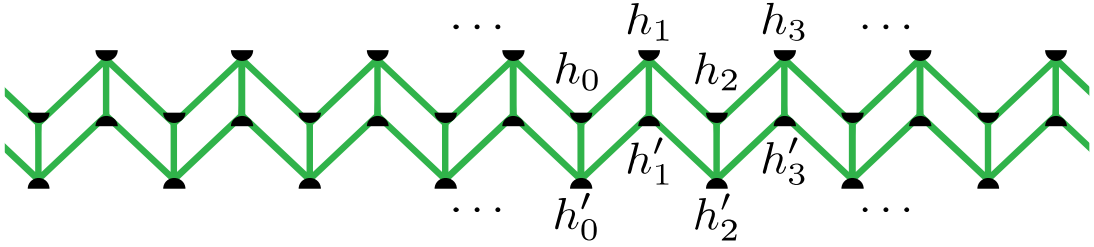


One type of defect for each object ϕ, with weight $\propto F_{\phi\rho}\begin{pmatrix} a & b' \\ b & a' \end{pmatrix}$

In the presence of the defect, the partition function is modified to

$$\mathcal{Z} = \sum_{\text{heights}} \prod_{\text{faces}} \diamond \times \prod_{\text{along defect}} \square$$

Defect-line creation operator: $\mathcal{D} =$



Two types of topological defects in Ising

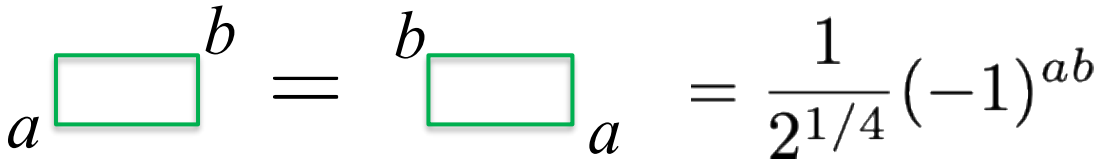
For Ising, fluctuating degrees of freedom on only half the sites:

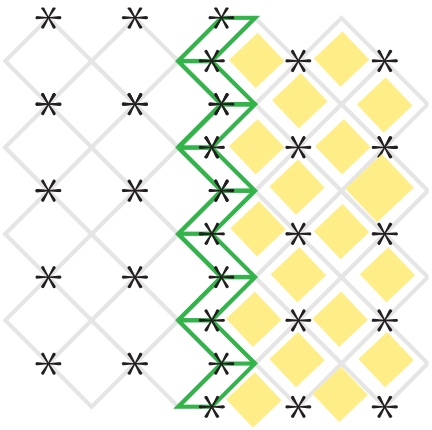


spin-flip defect:

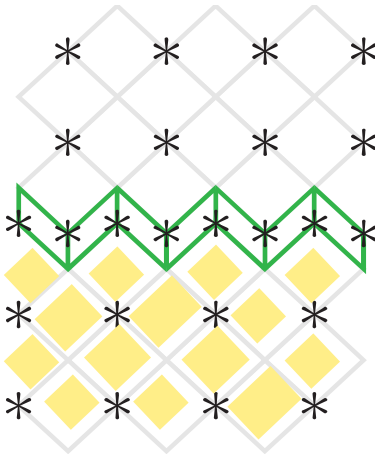


duality defect:





Duality defect changes sublattice.
Coupings on one side of duality defect are dual to those on the other!



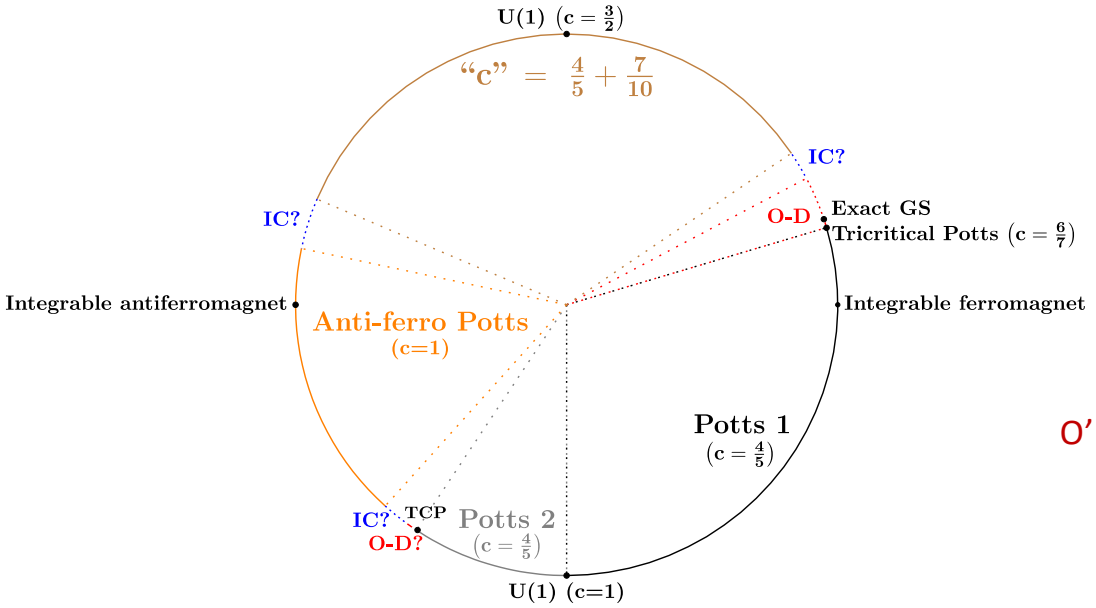
Many well-known models fit in this framework

Any ``categorical" lattice model has a family of topological defects!

Seems likely there exists at least at least one (often more than one) such lattice model for each RCFT.

Category of lattice model is typically a subcategory of the category describing RCFT.

Important to emphasise that generically, these lattice models are neither critical, nor integrable, even with spatially uniform couplings. Very rich phase diagrams, e.g.



O'Brien & Fendley 2019

Rydberg blockade

Rydberg-atom arrays provide a way of realizing all sorts of interesting phases in lattice models. The exceptional ability to tune interactions allows one to carefully examine these phases and the transitions between them.

In the Rydberg blockade, each site is effectively a two-state system that can be viewed as empty or as occupied by a hard-core boson. The blockade means that neighbouring (or more) sites cannot both be occupied.

Experiments on the chain have already revealed much interesting physics in strongly interacting systems. The golden chain has been realized!

arXiv:2601.16275v1

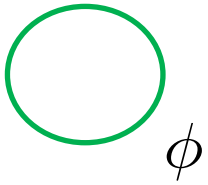
Experimental observation of conformal field theory spectra

Xiangkai Sun,[1,*] Yuan Le,[1,*] Stephen Naus,[1,*] Richard Bing-Shiun Tsai,[1,*] Lewis R. B. Picard,[1] Sara Murciano,[2] Michael Knap,[3,4] Jason Alicea,[1] and Manuel Endres[1,†]

Here we directly observe the energy excitation spectra of emergent CFTs at quantum phase transitions—recovering universal energy ratios characteristic of the underlying field theories[13–15]. Specifically, we develop and implement a modulation technique to resolve a Rydberg chain's finite-size spectra, variably tuned to quantum phase transitions described by either Ising or tricritical Ising CFTs. We also employ lo-

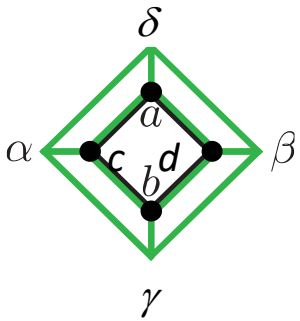
Use #1: Generalized K-W duality

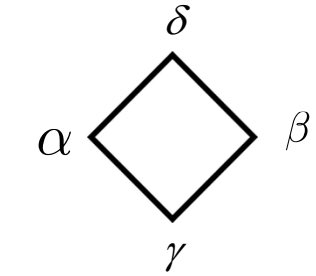
Automatically get identities between partition functions with different defect configuations

In a category, 

$= d_\phi$

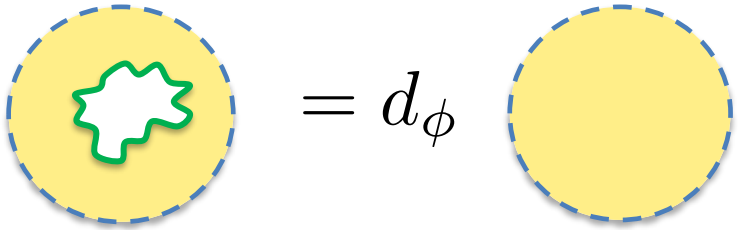
By either the abstract manipulations or using the explicit defect weights, find

$$\sum_{a,b,c,d}$$ 

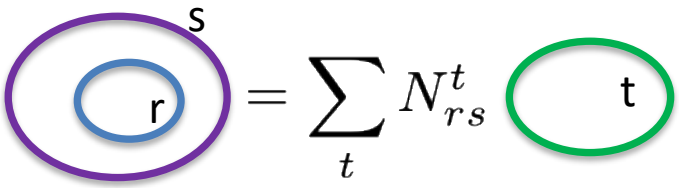
$= d_\phi$ 

Use commutation relations to move defect around. For any path, partition functions on a disc are related as

$$\mathcal{Z} = d_\phi Z$$



Fusing

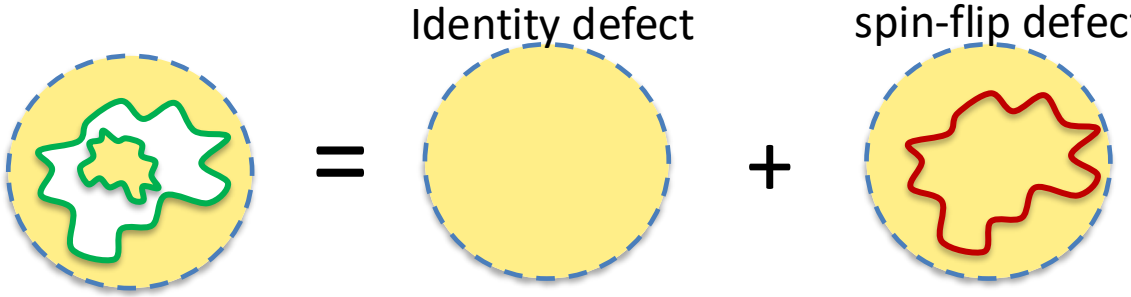


$$= \sum_t N_{rs}^t$$

so e.g. $\mathcal{D}_\alpha \mathcal{D}_\beta = \sum_\gamma N_{\alpha\beta}^\gamma \mathcal{D}_\gamma$

In Ising category (and CFT): $\mathcal{D}_{1/2}\mathcal{D}_{1/2} = 1 + \mathcal{D}_1$

Combine two duality defects:



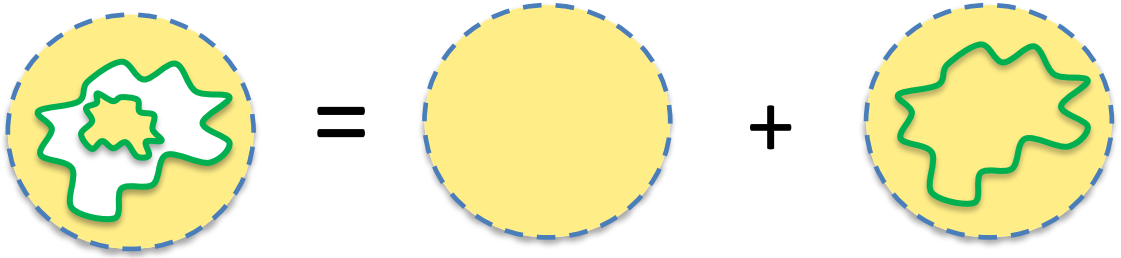
KW duality is not a symmetry – even at the critical point, it changes boundary conditions.

(At Ising critical point, in the Hamiltonian limit, with periodic b.c., you can mix KW duality with translation symmetry to get an actual symmetry.)

Seiberg and Shao 2023

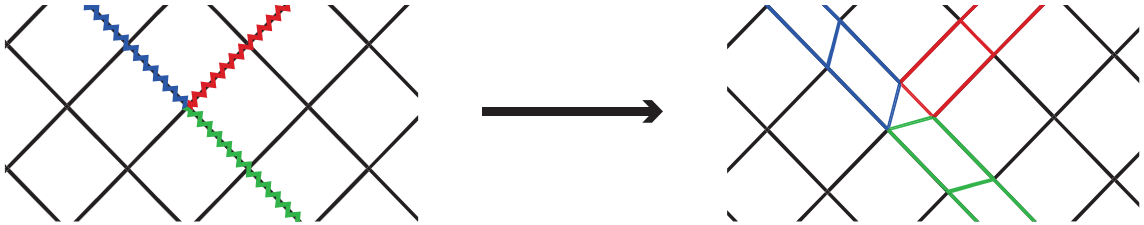
``Topological" symmetry for Fibonacci category is actually a symmetry (but is invertible)

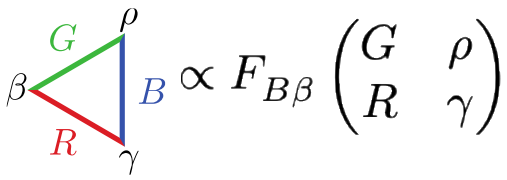
$$\mathcal{D}_1\mathcal{D}_1 = 1 + \mathcal{D}_1$$



Micro to macro: Defects themselves satisfy category rules!

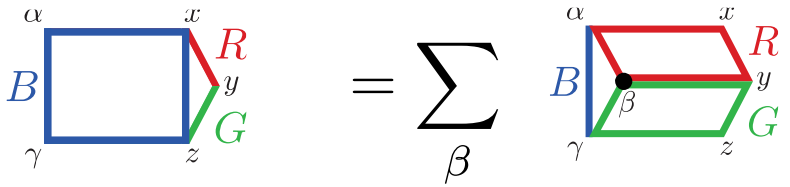
Defects can branch by defining a triangle vertex where three defects join:



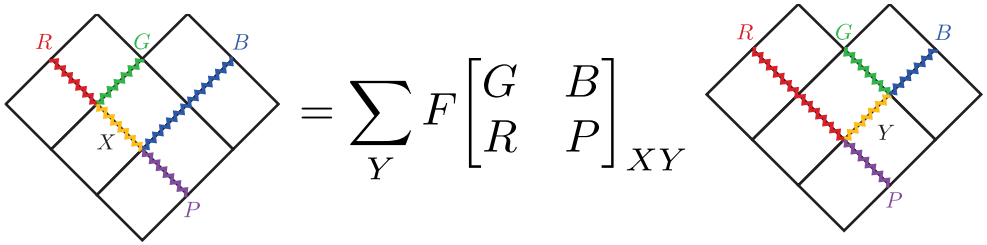


$$\propto F_{B\beta}\begin{pmatrix} G & \rho \\ R & \gamma \end{pmatrix}$$

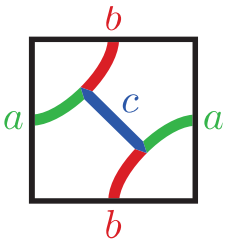
Derive the junction commutation relation



A little more work then gives



Can ignore lattice and do F moves on defects !

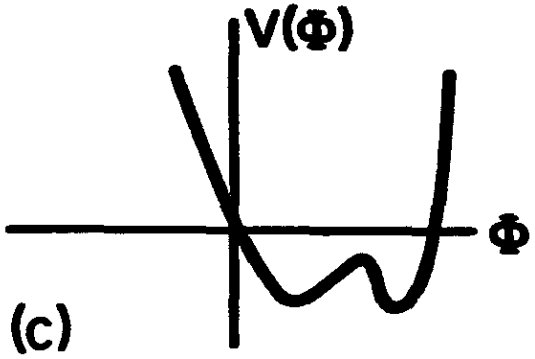


Use #2: Deriving exact degeneracies

THE SCALING REGION OF THE TRICRITICAL ISING MODEL IN TWO DIMENSIONS

Michael LÄSSIG, Giuseppe MUSSARDO* and John L. CARDY

A certain integrable massive field theory has effective potential with two asymmetric degenerate minima. Initial observation numerical, then from quantum-group algebra.



Lassig-Mussardo-Cardy 90;
Zamolodchikov 90; Smirnov 91

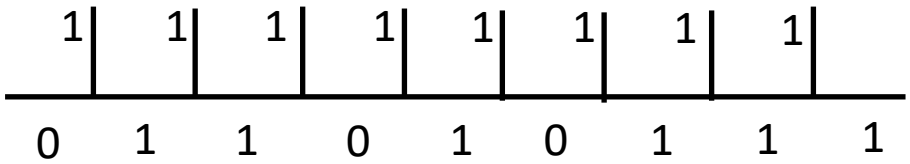
In the corresponding non-integrable spin chain (staggered ``golden chain''), this degeneracy follows immediately from the existence of a topological defect – no detailed analysis is required.

Aasen-Fendley-Mong 2020

Can understand degeneracies exactly on the lattice

FIbonacci: $1 \otimes 1 = 0 \oplus 1$ with $\rho = 1$

e.g.



Cannot have

$$h_j = 0,\ h_{j+1} = 0$$

If you think of 1 as being empty, and 0 as being a particle, get Rydberg blockade!

Build Hamiltonian using category, find

$$\mathcal{P}_j^{(0)} : \left|\ldots h_{j-1} h_j h_{j+1} \ldots\right\rangle \rightarrow \delta_{h_{j-1} h_{j+1}} \phi^{\frac{1}{2}(h_j + h'_j - h_{j-1} - h_{j+1})} \left|\ldots h_{j-1} h'_j h_{j+1} \ldots\right\rangle$$

$$\phi = \frac{1+\sqrt{5}}{2}$$

$$H = \sum_j c_j \mathcal{P}_j^{(0)}$$

recovering Hamiltonian limit of Baxter's critical integrable hard square model when all coefficients are the same

(aka A_4 RSOS/Temperley-Lieb Hamiltonian, aka golden chain)

Building from category tells you something new. Hamiltonian automatically commutes with ``topological symmetry", whose generator obeys

$$\mathcal{D}_1\mathcal{D}_1 = 1 + \mathcal{D}_1$$

$$\left[\mathcal{D}_1, \mathcal{P}_j^{(0)}\right] = 0$$

Feiguin et al 2007

Symmetry holds for any choice of coefficients in $H = \sum_j c_j \mathcal{P}_j^{(0)}$

If we take $c_{2k-1} = 0,\ c_{2k} < 0,$ get two exact ground states:

$\left|010101\ldots\right\rangle$ and $\left|1\widetilde{1}1\widetilde{1}1\widetilde{1}\ldots\right\rangle$ where $\left|\widetilde{1}\right\rangle = \phi\left|1\right\rangle + \left|0\right\rangle$

$\phi = \frac{1+\sqrt{5}}{2}$

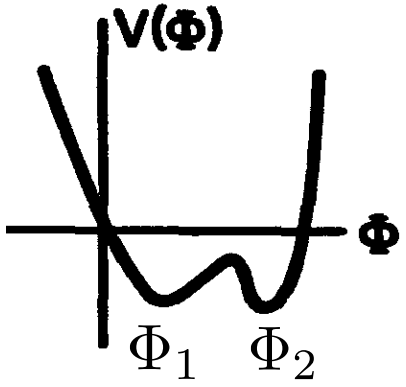
These two g.s. are related by $\mathcal{D}_1$ but no ordinary symmetry!

Now take $c_{2k} \neq 0$. Since symmetry persists, ground state multiplet can't go away without a phase transition.

Exact degeneracy must be preserved throughout gapped phase!

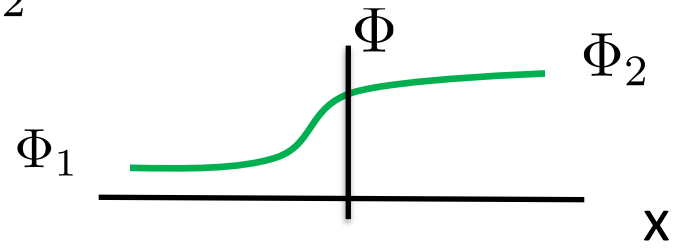
Still more:

Low-lying states in the potential

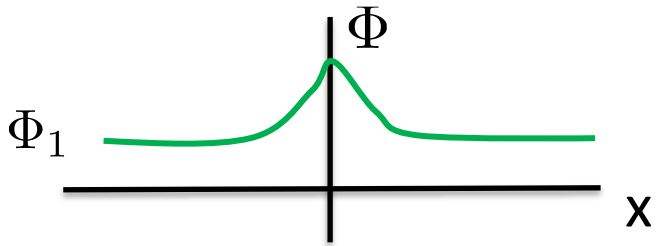


are comprised of:

Kinks interpolating between the two minima



Kink/antikink bound state only in shallow well:

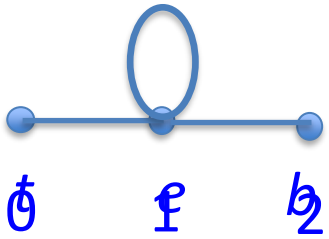


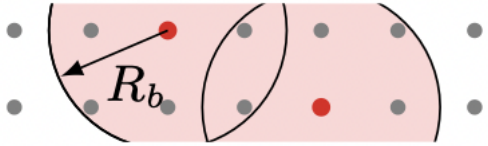
In spin chain, $\langle \text{kink} | \mathcal{D}_1 | \text{bound state} \rangle \neq 0$

The kink and the bound state are degenerate!

Integrable Rydberg-blockade ladder

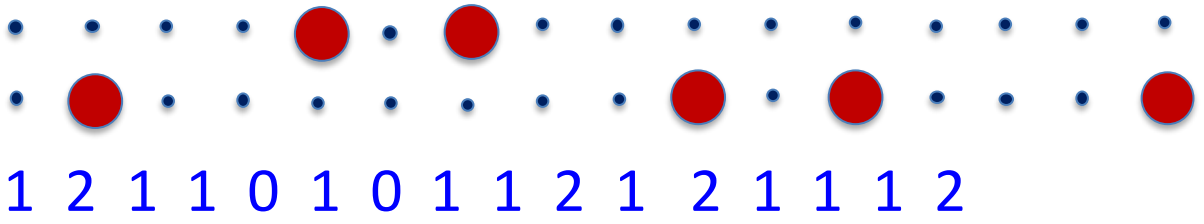
Using category, can construct ``height'' models with nice properties.
For $SU(2)_4$ - Rep(S_3) , degrees of freedom 0,1,2 obeying $1 \otimes 1 = 0 + 1 + 2$
adjacent heights on a chain must be adjacent on the diagram





Allowed configuration is 1211010112121112 -> ebeteteebebeeeb

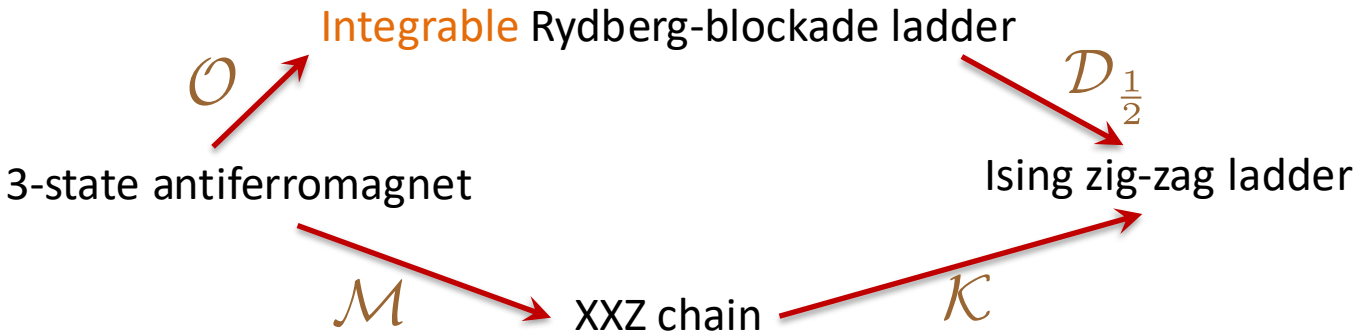
Interpret as a ladder: 0 is particle on top rung, 2 on bottom, 1 is empty rung



Rules mean at most one particle per square. The Rydberg blockade!

Use #3: Non-invertible, non-obvious mappings, e.g. $\mathrm{Rep}(S_3)/\,SU(2)_4$

Eck & Fendley 2023; see also Lootens et al



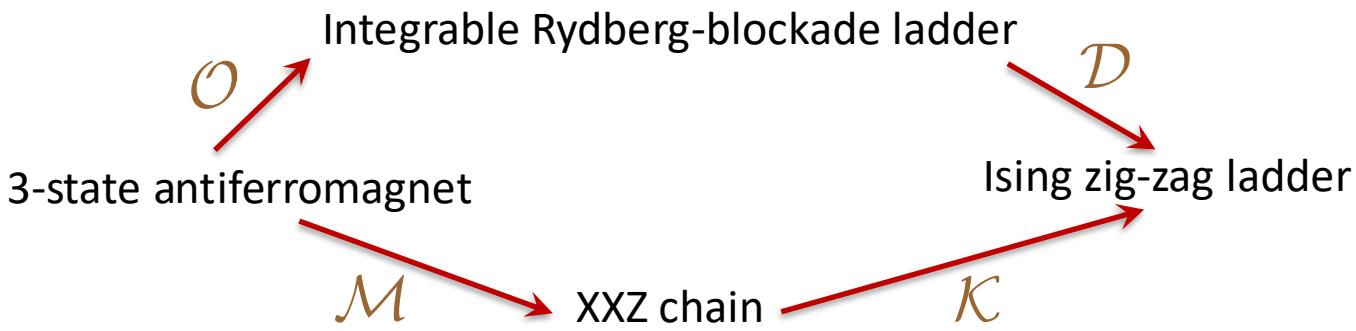
The Hamiltonians obey e.g. $\mathcal{K}\,H_{\mathrm{XXZ}} = H_{\mathrm{zig}}\,\mathcal{K}$ $\qquad$ $\mathcal{K}^\dagger\,H_{\mathrm{zig}} = H_{\mathrm{XXZ}}\,\mathcal{K}^\dagger$

The maps obey e.g. $\mathcal{K}\,\mathcal{K}^\dagger = 1 + \mathcal{F}$ $\qquad$ $\mathcal{K}^\dagger\,\mathcal{K} = 1 + (-1)^L \mathcal{F}_{\mathrm{zig}}$

$$\mathcal{D}_{\frac{1}{2}}\,\mathcal{O} = \mathcal{M}\,\mathcal{K}$$

XXZ chain is well understood, so gives new results about physics of other models

Non-invertible ``self-duality'' of the Rydberg ladder



The map $\mathcal{D}$ is associated with the spin-½ object of SU(2) . Since

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1 \qquad \Longrightarrow \qquad \mathcal{D}^\dagger \mathcal{D} = 1 + \widetilde{\mathcal{D}}$$

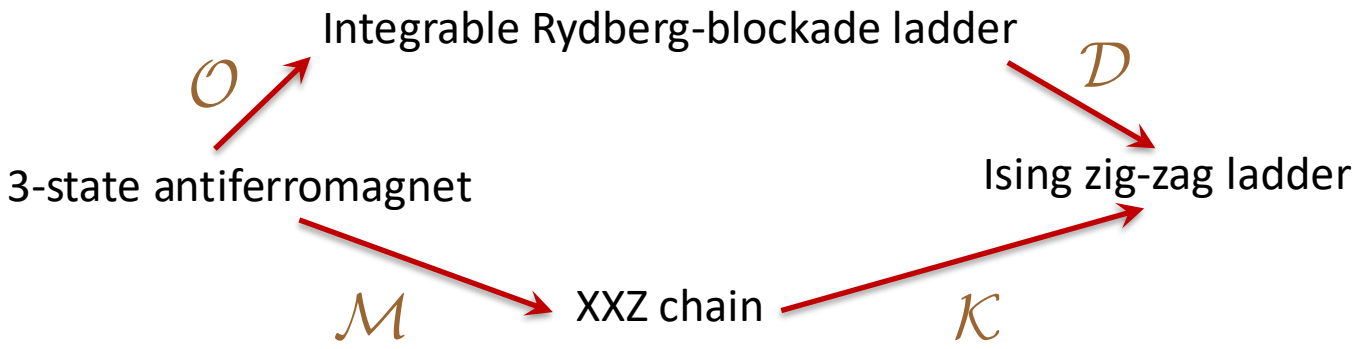
where $\widetilde{D}$ is associated with the spin-1 object. By construction, $\widetilde{\mathcal{D}} H_{\rm IRL} = H_{\rm IRL} \widetilde{\mathcal{D}}$

This symmetry generator is not invertible because

$$1 \otimes 1 = 0 \oplus 1 \oplus 2 \qquad \Longrightarrow \qquad \widetilde{\mathcal{D}}^2 = 1 + \mathcal{F} + \widetilde{\mathcal{D}}$$

Because it maps a Hilbert space to itself, we call this symmetry a self-duality.

Maps need not preserve U(1) symmetry.



Another non-invertible symmetry of the Rydberg-blockade ladder is a remnant of the *U*(1) symmetry of the XXZ chain (sometimes known as ``cosine" symmetry.

The reason is the map $\mathcal{K}$ annihilates the U(1) generator *Q*.

Instead, can map Q^2 via $\widetilde{Q} \equiv \mathcal{O}\mathcal{M}^\dagger Q^2 \mathcal{M}\mathcal{O}^\dagger$. Result commutes with $H_{\rm IRL}$

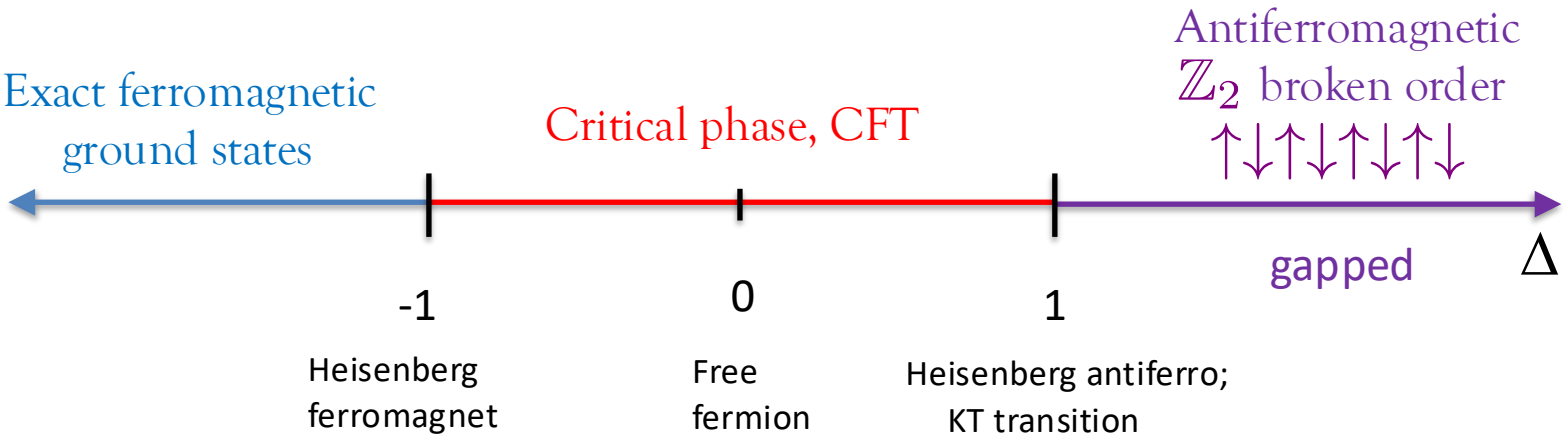
It acts non-trivially on only half of the Hilbert space. Explicitly,,

$$\mathcal{Q} = \tfrac{1}{2}\left(1+\mathcal{F}\right)\sum_{j=1}^{L}\sum_{k=0}^{L-1}\prod_{l=0}^{k}(-1)^{1-n^{-}_{j+l}}$$

This symmetry survives perturbation!

The physics of the integrable line

Phase diagram of XXZ



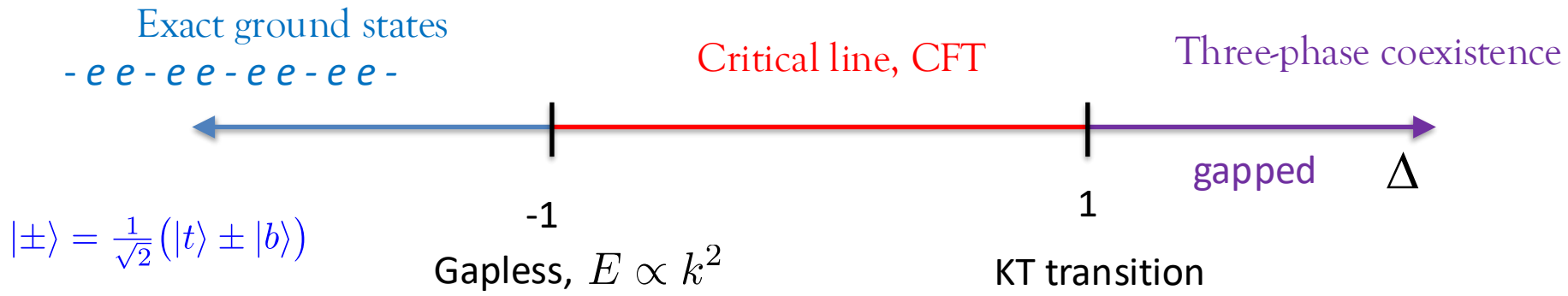
The effective field theory of XXZ is a free massless boson of radius R, where

$$\Delta = -\cos\frac{\pi}{2R^2} \qquad R \geq \frac{1}{\sqrt{2}}$$

Critical ground state survives all mappings: all models are critical for $-1 \leq \Delta \leq 1$

``Equivalent'' models are all c =1 CFTs, but not the same ones!

Phase diagram of integrable Rydberg-blockade ladder



We thus have located an exact critical line in the Rydberg-blockade ladder!

Its effective CFT is a *c* =1 orbifold with $\Delta = -\cos\frac{9\pi}{2R^2}$

As $\Delta \to \infty$, three ground states are

e e e e e e e e e e e e e
e + e + e + e + e + e +
+ e + e + e + e + e + e

No conventional symmetry relates them, but self-duality does.

Three ground states for all Δ > 1 !

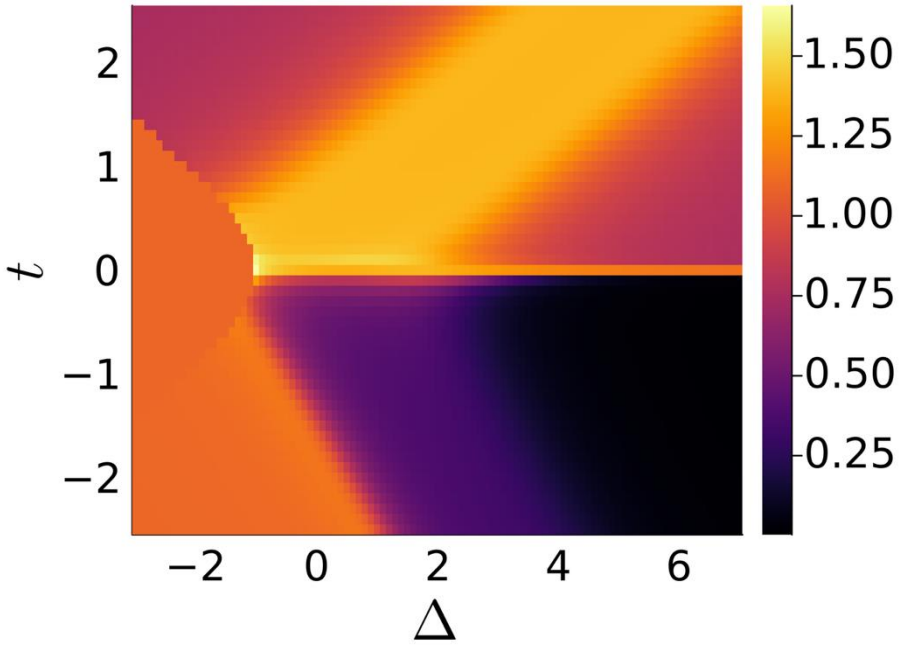
Off the integrable line

$$H = \sum_{j=1}^{L} \Big((1-w)(p_j + p_j^\dagger) + (1+2w) s_{j-1} s_{j+1} + (\Delta - 2t) n_j^- + (\Delta + t) \left(n_{j-1}^e - n_{j+1}^e \right)^2 \Big)$$

p_j annihilates a plus state, s_j swaps $+ \leftrightarrow -$, and $n_j^{(a)}$ are number operators

Deformation preserves not only the $\mathbb{Z}_2 \times \mathbb{Z}_2$ but the non-invertible symmetry!

Half-cut entanglement entropy at $w = 0$ from DMRG



Integrable line is $t = w = 0$

Lots of ordered phases with spontaneously broken translation symmetry

Two ground states:

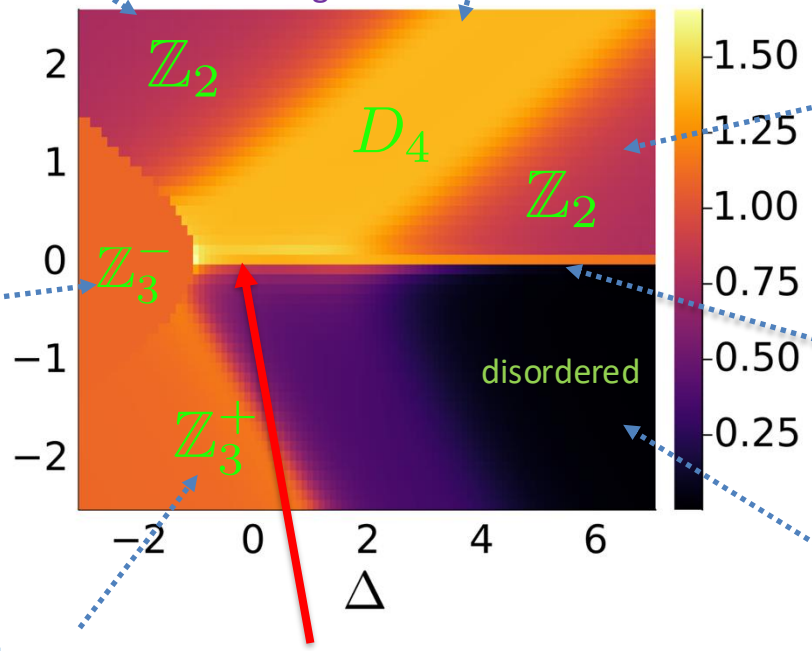
$e - e - e - e - e - e -$

Ising order gives four ground states:

$e\ t\ e\ b\ e\ t\ e\ b\ e\ t\ \ldots$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ here also broken so dihedral symmetry D_4

Again at $w = 0$



Ising disorder gives two ground states:

$e + e + e + e + e + e +$

Three exact ground states:

$e\ e - e\ e - e\ e -$

Integrable line is $t = w = 0$

Phase co-existence line

Unique ground state (no order)

$e\ e\ e\ e\ e\ e\ e\ e\ e\ e\ e\ e$

Three ground states:

$e\ e + e\ e + e\ e +$

Critical line

Equivalent to PXP for

$$\Delta = -t \to -\infty$$

Two $\mathbb{Z}_3$ - broken phases! Non-invertible symmetry distinguishes them.

Eck and Fendley

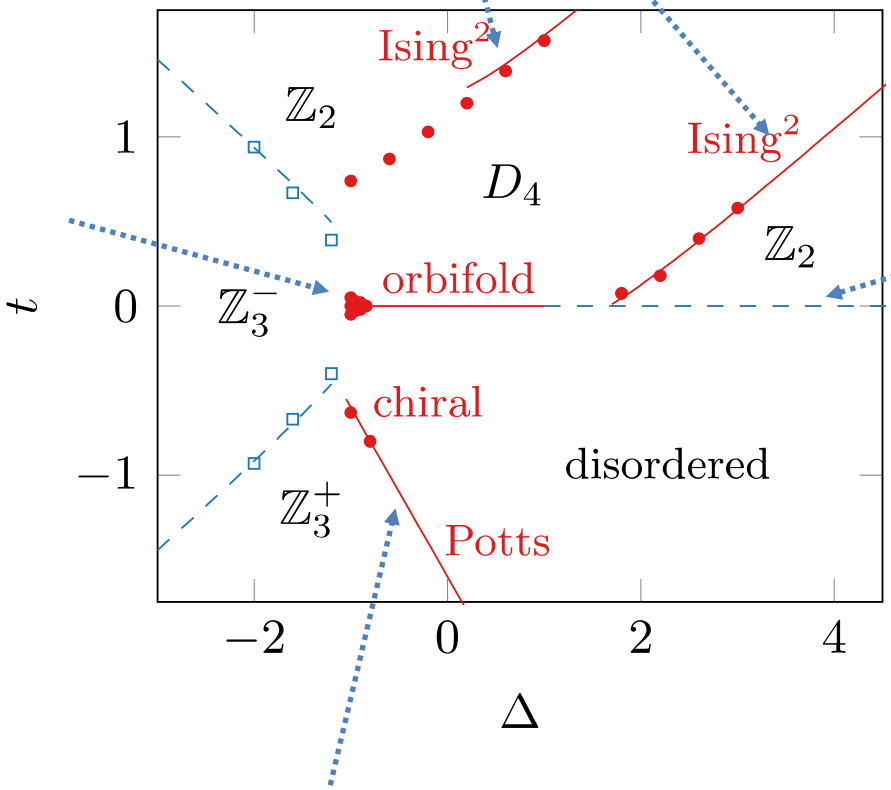
The transitions

Ising^2 transition from $t + \Delta$ large limit

First-order transitions: blue/dashed
second-order: red/solid

Lines from perturbations or exact; dots from ED

Small critical bubble!

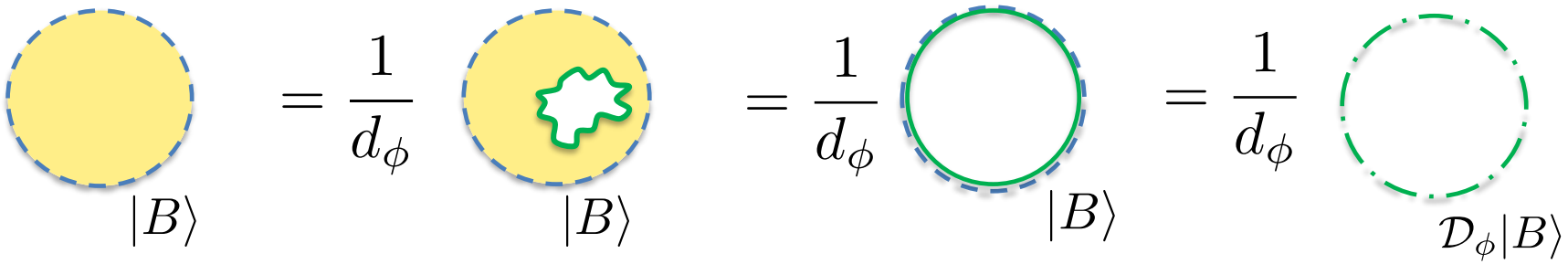


First-order transition at phase coexistence plane, including t=w=0 self-dual line.

Find both chiral and non-chiral Potts transitions

Use #4: Exact g-factor ratios from lattice

Aasen-Fendley-Mong 2020



Consider vector space of all configurations of spins/heights near edge. Each vector $|B\rangle$ corresponds to a boundary condition, e.g. $|\text{fixed up}\rangle = |+++++\ldots\rangle$

Acting with $\mathcal{D}_\phi$ absorbs the defect into edge, changing $|B\rangle \to \mathcal{D}_\phi|B\rangle$

$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi$$

Exact!!

e.g. duality defect in Ising changes b.c. as $\mathcal{D}_\sigma|\text{fixed up}\rangle = |\text{free}\rangle$

$$Z_{\text{free}}(K) = \sqrt{2}\, Z_{\text{fixed}}(\widehat{K})$$

where dual coupling is defined by $\sinh(2K)\sinh(2\widehat{K}) = 1$

For conformal boundary conditions, this ratio of partition functions is

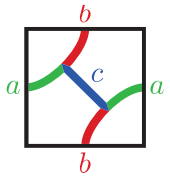
$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi = \frac{g_{\mathcal{D}_\phi|B\rangle}}{g_{|B\rangle}}$$

where $-T \ln g_{|B\rangle}$ is the boundary free energy. This Affleck-Ludwig g-factor is universal, and computable in RCFT.

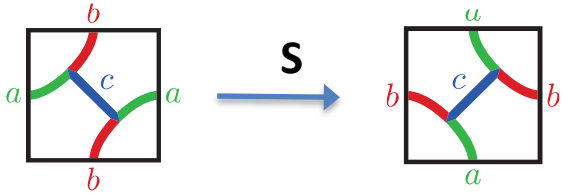
Calculation is direct and easy here! Moreover, gives precise lattice expressions for boundary states. Provides precise way of comparing lattice and CFT.

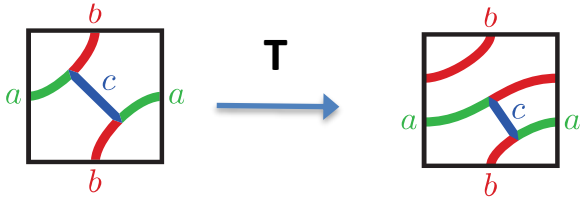
Another way to compare is to compute Dehn twist (i.e. modular transformation) in the presence of twisted boundary conditions (vertical topological defect). Yields conformal dimensions up to half-integer shift!

Use #4': exact CFT results from lattice from modular transformations

Basis elements for the toroidal partition functions are $Z^b_{ac} \equiv$ 

Under modular transformations:





Can use F-moves to manipulate pictures back to a canonical form:

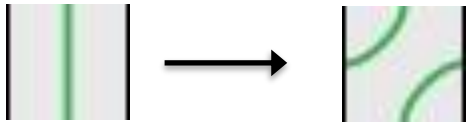
$$\mathbf{S}(Z^b_{ac}) = \sum_\gamma \left[F^{ba}_{ab}\right]_{c\gamma} Z^a_{b\gamma} \qquad \mathbf{T}(Z^b_{ac}) = \sum_\gamma \left[F^{ab}_{ba}\right]_{c\gamma} Z^b_{\gamma a}$$

Modular transformations are exact linear transformations of lattice partition functions..

Lattice to CFT

If you assume the lattice model has a continuum limit and becomes a CFT (sometimes true!) then a Dehn twist gives exact results for e.g. scaling dimensions. For example,

$\mathrm{T}^{(\varphi)}$:



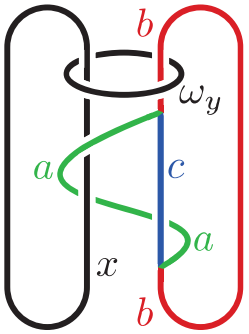
We whow that the lattice eigenvalues $e^{i2\pi t_\varphi}$ are related exactly to the spin of the corresponding CFT operator by

$$s_\varphi = \frac{n}{2} + t_\varphi \quad \text{for integer } n$$

Get 1/16 for Ising, 1/5 for Fibonacci!... Yields a strong constraint on any continuum limit.

An even stronger conjecture of Dave Aasen's:

$$Z_M = \sum_{ij} \bar{\chi}_i(\bar{\tau}) M_{ij} \chi_j(\tau) \qquad M^*_{xy}[Z^b_{ac}] \equiv \frac{e^{2\pi i h_a}}{d_y}$$



Use #5: Baxterization from fractional-spin conserved currents

- Bernard and Felder (and later Smirnov) defined non-local operators in that give conserved currents in several lattice models.

- Examples are fermion operator in the Ising, or parafermions in the 3-state Potts model. Cardy and collaborators found many more in geometric lattice models.

- Cardy et al also had a different philosophy. They did not require a priori that the Boltzmann weights satisfy the YBE. Requiring this conserved current exist then gives a linear condition for the Boltzmann weights. Solving it gives turns out to yield a solution of the full trilinear Yang-Baxter equation. Baxterization!

- By terminating the defects, category gives a natural and general way of defining such operators in local and geometric models and the condition they satisfy. The ensuing condition on the Boltzmann weights can be solved easily.

Fendley 2020

Use #6: 2d quantum Hamiltonians generalizing Kitaev honeycomb model

- The ``categorical" quantum Hamiltonians described above can be generalized to two dimensions by using fusion 2-categories.

  Inamura and Ohomori 2023

- Luckily in the cases of most interest, all one needs is a braided fusion category, i.e. what I described above plus braiding, i.e. allowing diagrams with over- and undercrossings.

- I-O show that the Ising category yields the Kitaev honeycomb model. The setup automatically gives local symmetries, the ``plaquette" conserved fluxes.

$$H^{\text{Kitaev}} = -J_x \sum_{a,b\in\text{x-link}} X_a X_b - J_y \sum_{a,c\in\text{y-link}} Y_a Y_c - J_z \sum_{b,d\in\text{z-link}} Z_b Z_d,$$

- Category gives a natural method general way of defining 2d models with such symmetries for any braided fusion category. Time-reversal-symmetry breaking is built in, so get promising candidates for chiral topological order!

  Eck and Fendley 2024

Conclusions

- One feature of these lattice calculations is their simplicity. Extract exact and universal results almost by definition
- Provides useful diagnostic tool for identifying lattice with continuum
- Naturally generalizes to higher dimensions -- story just beginning
- Connections to integrability profound, but not well understood
- Moral: if an interesting concept originates on the lattice, probably a good idea to keep thinking about the lattice